Left endpoint approximation and error bound

To approximate the definite integral $\int_a^b f(x)\,dx$, we can use left endpoint rectangles with a total of n rectangles. We denote this approximation L_n. The error in this approximation is

$$E_n = \left| \int_a^b f(x)\,dx - L_n \right|.$$

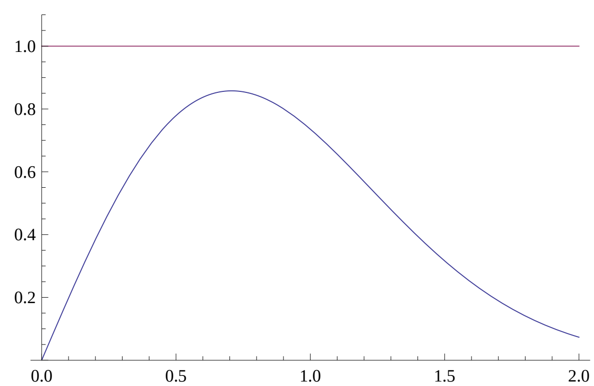
In class, we found an upper bound on this error. The upper bound B_n involves a, b, n, and some information about the derivative $f'(x)$. In particular, we need an upper bound M on $|f'(x)|$ for all x between a and b. The upper bound we found is

$$B_n = \frac{1}{2} M \frac{(b-a)^2}{n}.$$

Example: For $\int_0^2 e^{-x^2}\,dx$, the left endpoint approximation with four rectangles is

$$L_4 = \left(e^{-0^2} + e^{-0.5^2} + e^{-1.0^2} + e^{-1.5^2}\right)(0.5) = 1.1260$$

To get a bound on the error in this approximation, we first need a bound on $|f'(x)|$. With $f(x) = e^{-x^2}$, we compute $f'(x) = -2xe^{-x^2}$ so $|f'(x)| = 2xe^{-x^2}$ for x between 0 and 1. Below is a plot of $|f'(x)| = 2xe^{-x^2}$.



From this plot, we see that $M = 1$ is an upper bound for $|f'(x)| = 2xe^{-x^2}$. With a value for M in hand, we can compute

$$B_4 = \frac{1}{2}(1)\frac{(2-0)^2}{4} = 0.5.$$

So, the error in $L_4 = 1.1260$ is no bigger than 0.5. In other words, the exact value of the integral is somewhere between $1.1260 - 0.5 = 0.6260$ and $1.1260 + 0.5 = 1.6260$. Given that the error bound is in the tenths, we should round to tenths and report that the exact value is between 0.6 and 1.7. (Note that we have rounded the lower value down and the higher value up so that we are still guaranteed that the exact value is between the two rounded values.)

The next page has a few problems using this idea.

Problems

1. For $\displaystyle\int_0^2 e^{-x^2}\,dx$, compute an approximation using 8 left endpoint rectangles and determine a bound on the error for this approximation.

2. For $\displaystyle\int_0^2 e^{-x^2}\,dx$, compute an approximation using 8 *right* endpoint rectangles and determine a bound on the error for this approximation. Note: The error bound given above also applies to right endpoint approximations.

3. For $\displaystyle\int_0^2 e^{-x^2}\,dx$, determine the number of rectangles needed to get a left endpoint approximation within a tolerance of 0.01.

4. For $\displaystyle\int_1^3 \sin(x^2)\,dx$, compute an approximation using 10 left endpoint rectangles and determine a bound on the error for this approximation.